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A COMPARISON ON ERROR BOUNDS AND ERROR TERMS FOR LOWER AND HIGHER ORDER LAGRANGE'S POLYNOMIAL INTERPOLATION (EQUALLY SPACED NODES) BY USING EXPONENTIAL FUNCTION

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Abstract

In this paper, a comparative study is performed on error bounds and error terms for lower and higher order Lagrange polynomial approximation (equally space nodes). The work has focused the different error terms for dissimilar order Lagrange polynomial such as linear, Quadratic, Cubic, Fourth and Fifth orders. The error bounds and error terms are presented by an appropriate example of exponential function. It is found that the higher order Lagrange polynomial approximation is better than the lower order. The comparative results of exponential function and their Lagrange polynomial approximations are illustrated graphically by using MATLAB 7.0.

Keywords: Lagrange polynomial, Error terms, Error bounds and Numerical process.

Introduction

The polynomial approximations of the library function such as sin(x), cos(x), or e^x are evaluated by using computer software (John H. Mathews, 2001**)**. The analysis of computer errors and the other sources of error in the numerical method is a critically important part of the study of numerical analysis (Burden and Fairs, 2004). When numerical analysis is accomplished, there are several possible sources of error arises. Some are avoidable, some are not. For example data conversion and round off errors cannot be avoided, but a human error can be eliminated. It is essential to know how error arises, how they grow during the numerical process and how they influence the accuracy of a solution (Sastry, 2003).

Governing Equation

Let a polynomial $P(x)$ of degree N will be constructed which passes through $N + 1$ points. The polynomial $P(x)$ can be used to approximate $f(x)$ over the entire interval [a, b]. To simplify the error function $R(x) = f(x) – P(x)$, we have to know $F^{N+1}(x)$ and a bound for its magnitude, that is, M = max {$|f^{N+1}(x)|$: for $a \le x \le b$ }. Let us briefly mention how to evaluate the polynomial $P(x): P(x) = a_N x^N + a_{N-1}x^{N-1} + \ldots + a_2 x^2 + a_1 x + a_0$. The

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French Mathematician Joseph Louis Lagrange is used a slightly difference method to find this polynomial. He noticed that for two points (x_0, y_0) and (x_1, y_1) and it was written as

$$y = P_1(x) = \frac{(x - x_1)}{(x0 - x_1)} y_0 + \frac{(x - x_0)}{x_1 - x_0} y_1 \tag{1}$$

The quotients in (1.1) are denoted by

$$L_{1,0}(x) = \frac{x - x_1}{x_0 - x_1} \text{ and } L_{1,1}(x) = \frac{x - x_1}{x_1 - x_0} \tag{2}$$

The term $L_{1,0}(x)$ and $L_{1,1}(x)$ are called the Lagrange coefficient polynomials based on the nodes x_0 and x_1. Using this notations (1) can be written in summation form:

$$P_i(x) = \sum_{k=0}^{i} y_k L_{i,k}(x) \tag{3}$$

The generalization of (3) is the construction of a polynomial $P_N(x)$ of degree at most N that passes through the $N + 1$ points (x_0, y_0), (x_1, y_1), ..., (x_N, y_N) and has the form:

$$P_N(x) = \sum_{k=0}^{i} y_k L_{N,k}(x) \; ;$$

where $L_{N,k}(x)$ is the Lagrange coefficient polynomial based on these nodes:

$$L_{N,k}(x) = \frac{(x - x_0)...(x - x_{k-1})(x - x_{k+1})...(x - x_N)}{(x_k - x_0)...(x_k - x_{k-1})(x_k - x_{k+1})...(x_k - x_N)}$$

$$= \frac{\prod_{\substack{j=0 \\ j \neq k}}^{N} (x - x_k)}{\prod_{\substack{j=0 \\ j \neq k}}^{N} (x_j - x_k)} \tag{4}$$

In this paper, we show that if a function is described in terms of Lagrange polynomial approximation, the error terms will decreases, as the order of polynomial is increase.

Lagrange Polynomial Approximation Theorem

Assume that $f \in C^{n+1}[a,b]$ and that $x_0, x_1, ..., x_N \in [a,b]$ are N + 1 nodes. If $x \in [a,b]$, then $f(x) = P_n(x) + R_N(x)$ (5)

where $P_N(x)$ is a polynomial that can be used to approximate $f(x)$.

$$f(x) \approx P_N(x) = \sum_{k=0}^{N} f(x_k) L_{n,k}(x) \tag{6}$$

The error term $R_N(x)$ has the form

$$R_n(x) = \frac{(x-x_0)(x-x_1)...(x-x_N)f^{N+1}(c)}{(N+1)!} \quad (7)$$

for some value $c = c(x)$ that lies in the interval $[a, b]$

Error analysis by accuracy theorem

The quality of $P_n(x)$ depends on the choice of the $(n + 1)^{th}$ derivative of (7) and on x. We see in the above theorem in the error formula

$$f(x) = Pn(x) + \frac{f_n^{n+1}}{(n+1)!}(x-x_0)(x-x_1)...(x-x_n);$$

where $a = x_0 \le x \le x_n = b$. If we have a bound on this derivative, then we can compute a bound on the error. To illustrate this in a practical way, suppose $|f_{n+1}| \le M_{n+1}$ for $x \in [a,b]$.It follows that for any $z \in [a,b]$ we have

$$|f(z) - P_n(z)| \le \frac{M_{n+1}}{(n+1)!} \max |(x-x_0)(x-x_1)...(x-x_n)| \quad (8)$$

If we base the interpolant on the equally spaced points

$$x_{i-1} = a + \frac{b-a}{(n+1)}(i-1), \quad i = 1,2...(n+1)$$

Putting the values x_0, x_1, x_2, ..., x_n in (8) then

$$|f(z) - P_n(z)| \le \frac{M_{n+1}}{(n+1)!}\left(\frac{b-a}{n}\right) \max a \le x \le b \left|\left(\frac{x-a}{b-a}\right) n. \left(\frac{x-a}{b-a}n - 1\right) ... \left(\frac{x-a}{b-a}n - n\right)\right|$$

$$\le M_{n+1}\left(\frac{b-a}{n}\right)^{n+1} \max \left|\frac{s(s-1)(s-2)...(s-n)}{(n+1)!}\right| ; \text{ where } s = \frac{(x-a)n}{b-a}$$

It can be shown that the max is no longer then $\frac{1}{4(x+1)}$, from which we conclude that

$$|f(z) - P_n(z)| \le \frac{M_{n+1}}{4(n+1)}\left(\frac{b-a}{n}\right)^{n+1}$$

Thus, if a function has ill-behaved higher derivatives then the quality of the polynomial interpolant may actually decrease as the degree increases.

Error Estimate and Error Bound for Lagrange Polynomial Interpolation (Equally space nodes)

It is important to understand the nature of the error term when the Lagrange polynomial is used to approximate a continuous function $f(x)$. It is similar to the error term for the Taylor polynomial, except that the factor $(x - x_0)^{N+1}$ is replaced with the product $(x - x_0)$ $(x - x_1)$ ... $(x - x_N)$. This is expected because interpolation is exact at each the $N + 1$ nodes x_k, where we have $R_N(x_k) = f(x_k) - P_N(x_k) = y_k - y_k = 0$ for $k = 1, 2, ..., N$. Assume that

$f \in C^{n+1}[a,b]$ and that $x_0, x_1, ..., x_N [a,b]$ are $N + 1$ nodes. If $x \in [a,b]$, then

$$f(x) = P_n(x) + R_n(x) \tag{9}$$

where $P_N(x)$ is a polynomial that can be used to approximate $f(x)$.

$$f(x) \approx P_n(x) = \sum_{k=0}^{N} f(x_k) L_{n,k}(x) \tag{10}$$

The error term $R_N(x)$ has the form

$$R_n(x) = \frac{(x - x_0)(x - x_1)...(x - x_N) f^{N+1}(c)}{(N+1)!} \tag{11}$$

for some value $c = c(x)$ that lies in the interval $[a, b]$

Assume that $f(x)$ is define on $[a, b]$ that contains the equally spaced nodes $x_k = x_0 + hk$ where h is the height of the interval. Suppose that $f(x)$ and the derivatives up to the order $N + 1$ are continuous and bounded on the subintervals $\{x_0, x_1], [x_0, x_2], ..., [x_0, x_n]$ respectively. That is,

$$\left| f^{N+1}(x) \right| \le M_{n+1} \text{ for } x_0 \le x \le x_N \tag{12}$$

Using (11) and (12) we obtain the following error terms and error bounds for respective polynomials:

Table 1. Error terms and Error bounds for different order polynomial

Lagrange Polynomial	Error Term	Error Bound
$P_1(x)$	$R_1(x)$	$R_1(x) \le \frac{h^2 M_2}{8}$
$P_2(x)$	$R_2(x)$	$R_2(x) \le \frac{h^3 M_3}{9\sqrt{3}}$
$P_3(x)$	$R_3(x)$	$R_3(x) \le \frac{h^4 M_4}{24}$
$P_4(x)$	$R_4(x)$	$R_4(x) \le \frac{\sqrt{4750 + 290\sqrt{145}}}{3000}$
$P_5(x)$	$R_5(x)$	$R_4(x) \le \frac{(10 + 7\sqrt{7})}{1215} h^6 M_6$

Results and Discussion

To demonstrate the comparison between higher order Lagrange polynomials and lower order Lagrange polynomials we consider the function $y = f(x) = ex$ over the interval [0.0, 1.8].we construct various polynomials for equal space nodes.

First order Lagrange Polynomial

Using the node $x_0 = 0.0$, $x_1 = 1.8$ and $y_0 = e_0 = 1$, $y_1 = e^{1.8} = 6.04965$ in (3) and (4) we get first order Lagrange polynomial $P_1(x) = 2.80536\,x + 1$ also $f^2(x) = e^x$ hence

$\left|f^2(1.8)\right| = e^{1.8} = 6.04965$ so that $M_2 = 6.04965$ (1

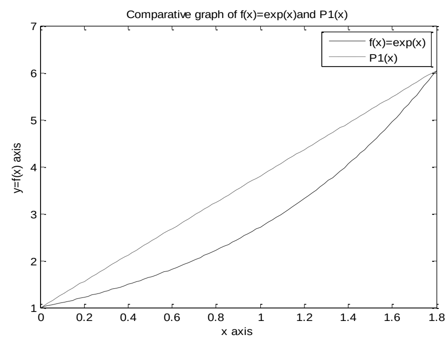
Therefore for $P_1(x)$ the spacing of the nodes is $h = 1.8$ and its error bound is

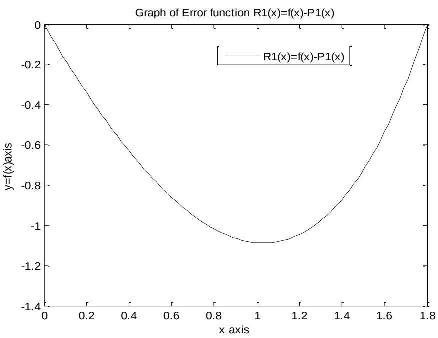
$$R_1(x) \le \frac{h^2 M_2}{8} = \frac{(1.8)^2\ 6.04965}{8} = 2.45011 \tag{14}$$

Also $\left|R_1(0.6)\right| = \left|e^{.6} - p_1(.6)\right| = \left|1.82212 - 2.68322\right| = 0.8611$

So the error bound 2.45011 from (14) is not reasonable. The graph of the error function $R_1(x) = ex - P_1(x)$ is shown in the following figure:

Relative graphs for the linear polynomial $P_1(x)$





Second order Lagrange polynomial

Using nodes $x_0 = 0.0$, $x_1 = 0.9$, $x_2 = 1.8$ and $y_0 = e0 = 1$, $y_1 = e0.9 = 2.4896$, and $y_2 = 6.04965$ in (3) and (4) we get second order Lagrange polynomial $P_2(x) = 1.31509x_2 + 0.43819x + 1$

Also $f^3(x) = e^x$ and $|f^3(1.8)| = e1.8 = 6.04965$ so that $M_3 = 6.04965$

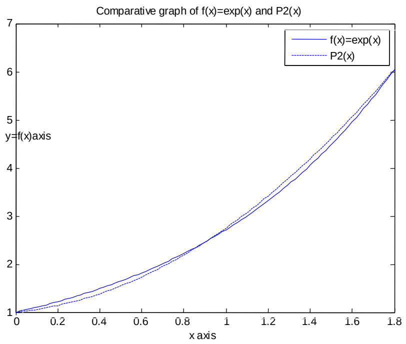
Therefore for $P_2(x)$ the spacing of the nodes is $h = 0.9$ and its error bound is

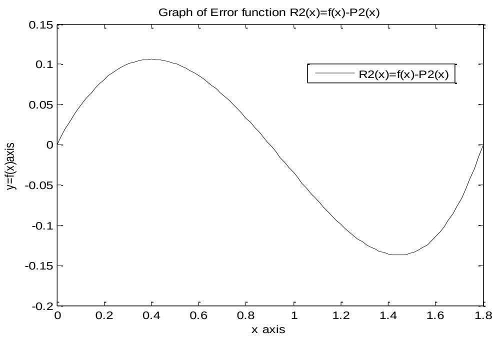
$$R_2(x) \le \frac{h^3 M_3}{9\sqrt{3}} = \frac{(0.9)^2 \times 6.04965}{9\sqrt{3}} = 0.28291 \tag{15}$$

Also $|R_2\,(1.2)| = |e^{1.2} - P_2(1.2)| = |3.32012 - 3.41956| = 0.09944$

So the error bound 0.28291 from (15) and error term 0.09944 is not closed, therefore it is not reasonable. The graph of the error function $R_2(x) = e^x - P_2(x)$ is shown in the following figure:

Relative graphs for the Quadratic Polynomial $P_2(x)$





Third order Lagrange polynomial

Using nodes $x_0 = 0.0$, $x_1 = 0.6$, $x_2 = 1.2$, $x_3 = 1.8$ and $y_0 = e^0 = 1$, $y_1 = e^{0.6} = 1.82212$,

$y_2 = e^{1.2} = 3.32012$, $y_3 = e^{1.8} = 6.04965$ in (3) and (4) we obtain the third order Lagrange polynomial $P_3(x) = 0.42874\, x^3 + 0.16698\, x^2 + 1.11566\, x + 1$.

Also $f^4(x) = ex$, $|f^4(1.8)| = e^{1.8} = 6.04965$ so that $M_4 = 6.04965$ (16)

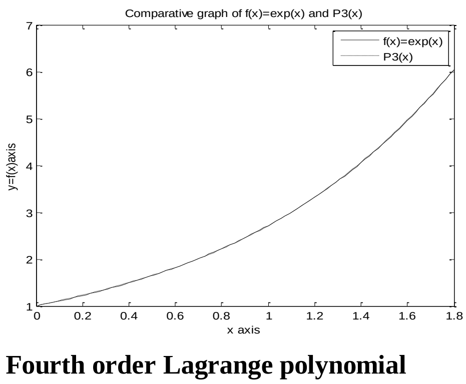
Therefore for $P_3(x)$ the spacing of the nodes is $h = 0.6$ and its error bound is

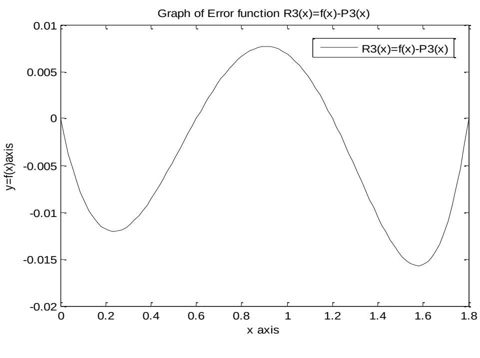
$$R_3(x) \le \frac{h^4 M_4}{24} = \frac{(0.6)^2 \times 6.04965}{24} = 0.03267 \qquad (17)$$

Also $|R_3(1.5)| = |P_3(1.5)| = |4.48469 - 4.49619| = 0.0145$

So the error bound 0.03267 from (17) and error term 0.0145 are not closed therefore, it is not reasonable. The graph of the error function $R_3(x) = e^x - P_3(x)$ is shown in the following figure:

Relative graphs for the Cubic Polynomial $P_3(x)$





Fourth order Lagrange polynomial

Now using nodes $x_0 = 0.0$, $x_1 = 0.45$, $x_2 = 0.9$, $x_3 = 1.35 = 18$ and

$y_0 = e^0 = 1$, $y_1 = e^{0.45} = 1.56831$, $y_2 = e^{0.9} = 2.4596$, $y_3 = e^{1.35} = 3.85743$, $y_4 = e^{1.8} = 6.04965$

in (3) and (4) we obtain fourth order Lagrange polynomial

$P_4 = 0.10598x^4 + 0.04965x^3 + 0.58036x^2 + 0.98204x + 1.0000$

Also $f^5(x) = ex$, $|f^5(1.8)| = e^{1.8} = 6.04965$ so that $M_5 = 6.04965$

Therefore for $P_4(x)$ the spacing of the nodes is $h = 0.45$ and its error bound is

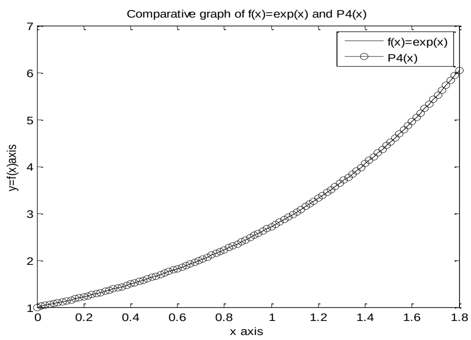
$$R_4(x) \le \frac{\sqrt{4750+290\sqrt{145}}}{3000} h^5 M_5 = \frac{(90.78581)(0.45)5 \times 6.04965}{3000} = 0.00338$$

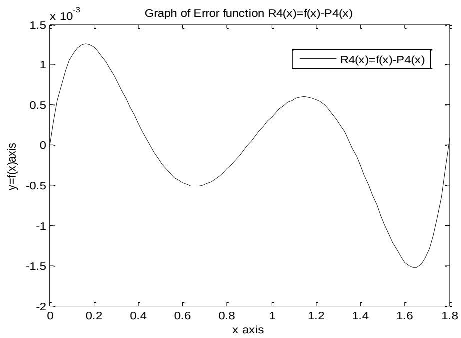
$$R_4(x) \le 0.00338 \qquad (18)$$

Also $|R_4(1.5)| = |e^{1.5} - P_4(1.5)| = |4.48169 - 4.48263| = 0.00094$

So here we also see that, the error bound 0.00338 from (18) and error term 0.00094 are not closed therefore, it is not so reasonable. The graph of the error function $R_4(x) = e^x - P_4(x)$ is shown in the following figure.

Relative graphs for Fourth order polynomial $P_4(x)$





Fifth order Lagrange polynomial

Now using nodes $x_0 = 0.0$, $x_1 = 0.36$, $x_2 = 0.72$, $x_3 = 1.08$, $x_4 = 1.44$, $x_5 = 1.8$ and

$y_0 = e^0 = 1$, $y_1 = e^{0.36} = 1.43333$, $y_2 = e^{0.72} = 2.05443$, $y_3 = e^{1.08} = 2.94468$, $y_4 = e^{1.44} = 4.2207$, $y_5 = e^{1.8} = 6.04965$ in (1.3) and (1.4), fifth order

Lagrange polynomial $P_5(x) = 0.02108x^5 + 0.01154x^4 + 0.19746x^3 + 0.46x^2 + 1.00255x + 1.00001$

Also $f^6(x) = e^x$, $|f^6(1.8)| = e^{1.8} = 6.04965$ So that $M_6 = 6.04965$

Therefore for $P_5(x)$ the spacing of the nodes is h = 0.36 and its error bound is

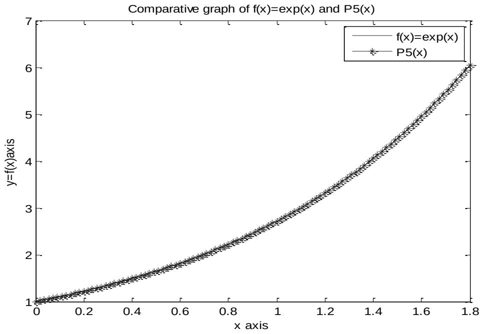
$$R_2(x) \le \frac{107\sqrt{7}}{1215} h^6 M_6 = \frac{28.52026\,(0.36)6 \times 6.04965}{1215} = 0.00031$$

$$R_2(x) \le 0.00031 \qquad (19)$$

Also $|R_5(1.5)| = |e^{1.2} - P_5(1.5) = |3.32012 - 3.32050| = 0.00039$

But here we confidently comment that, the error bound 0.00031 from (1.27) and estimate error 0.00039 are very much closed therefore, it is reasonable. The graph of the error function $R_5(x) = e^x - P_5(x)$ is shown in the following figure:

Relative graphs for Fifth order polynomial $P_5(x)$



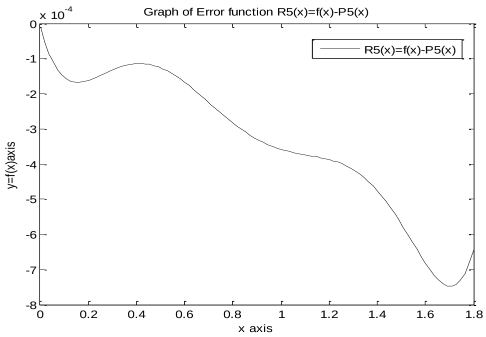


Table 2. Comparative error terms and error bounds for different order polynomials

Polynomials	Bounded Error	Error Term
First Order: $P_1(x)$	2.45011	0.8611
2nd Order: $P_2(x)$	0.28291	0.09944
3rd Order: $P_3(x)$	0.03267	0.0145
4th Order: $P_4(x)$	0.00338	0.00094
5th Order: $P_5(x)$	0.00031	0.00039

Conclusion

In this paper polynomial is expressed in terms of Lagrange polynomial approximation and some error arises. We have found that there are some specific bounds of this error for different order polynomial which are describe as before

[$P_1(x) = 0.8611$, $P_2(x) = 0.09944$, $P_3(x) = 0.0145$, $P_4(x) = 0.00094$, $P_5(x) = 0.00039$]

It is clear that the error terms decrease when the orders of polynomial increase with appropriate step length h. From this study we conclude that the higher order Lagrange polynomial approximation is better than the lower order approximation.

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