Journal of Science and Technology, 5(1): 133-141, January 2015

GENERALIZED TRAVELING WAVE SOLUTIONS OF SHALLOW WATER WAVE EQUATION USING EXP(–Φ(ξ))-EXPANSION METHOD

MD. ABDUS SALAM*, MD. MUSA MIAH and MD. BABUL HOSSAIN

Department of Mathematics, Mawlana Bhashani Science and Technology University, Tangail-1902, Bangladesh

Abstract

In this paper, some generalized traveling wave solutions of nonlinear shallow water wave equation have been found by means of the exp(–Φ(ξ))-expansion method. Assigning some special values for the parameters, solitary wave solutions can be originated from the generalized solutions easily. The validity of this method is checked by the solutions of the equation. It is established that the exp(–Φ(ξ))-expansion method offers a further influential mathematical tool for constructing traveling wave solutions of nonlinear evolution equations in mathematical physics.

Keywords: Nonlinear evolution equations, wave solutions, wave equation

Introduction

Nonlinear evolution equations are widely used as models to describe complex physical phenomena in various fields of sciences, especially in fluid mechanics, solid state physics, plasma physics, plasma wave and biology. One of the basic physical problems for those models is to obtain their travelling wave solutions. Particularly, various methods have been utilized to explore different kinds of solutions of physical models described by nonlinear partial differential equations (PDEs). However, in the recent years, a variety of effective analytical methods have been developed considerably to be used for nonlinear PDEs such as the homogeneous balance method (Wang *et al.*, 1996), the variational iteration method (He, 1999), the tanh-method (Zayed *et al*., 2004), (G'/G)-expansion method (Islam *et al*., 2013), modified simple equation method (Jawad *et al*., 2010; Salam, 2012), the exp-function method (Bakir, 2008), and so on.

Recently, a new powerful technique called exp(–Φ(ξ))-expansion method (Khan *et al*., 2013) has been developed for a reliable treatment of nonlinear wave equations. This method is straightforward, concise and capable of producing new applications. The organization of the paper is as follows: Firstly, a brief account of the exp(–Φ(ξ))-expansion method is given. Secondly, this method is applied to find exact traveling wave solutions of the equation. Thirdly, some graphs of the solutions are plotted and finally concluding remarks are presented.

* Corresponding author: salam.a.math03@gmail.com

Methodology

We consider a general nonlinear PDE in the form

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots\ldots\ldots \ldots\ldots\ldots \ldots\ldots) = 0, \tag{1}$$

The main steps of the exp(–Φ(ξ))-expansion method are given below:

Step-1: Using the transformation $u(x,t) = u(\xi)$, where $\xi = x + \omega t$ (ω =const.), we can rewrite (1) as the following nonlinear ODE:

$$Q(u, \omega u', u', \omega^2 u'', \omega u'', u'', \ldots\ldots\ldots\ldots\ldots\ldots\ldots) = 0 \tag{2}$$

Step-2: We consider a solution of (2) of the form

$$u(\xi) = \sum_{i=0}^{m} a_i (\exp(-\Phi(\xi)))^i, \tag{3}$$

where a_i are constants, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in (2), and $\Phi = \Phi(\xi)$ satisfies the equation:

$$\Phi'(\xi) = \exp(-\Phi(\xi)) + \mu \exp(\Phi(\xi)) + \lambda \tag{4}$$

Eq. (4) gives the following solutions:

(i) When $\lambda^2 - 4\mu > 0, \mu \neq 0$

$$\Phi(\xi) = \ln\left(\frac{-\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + E)\right) - \lambda}{2\mu} \right).$$

(ii) When $\lambda^2 - 4\mu < 0, \mu \neq 0$

$$\Phi(\xi) = \ln\left(\frac{\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + E)\right) - \lambda}{2\mu} \right).$$

(iii) When $\lambda^2 - 4\mu > 0, \mu = 0, \lambda \neq 0$

$$\Phi(\xi) = -\ln\left(\frac{\lambda}{\exp(\lambda(\xi + E)) - 1} \right).$$

(iv) When $\lambda^2 - 4\mu = 0, \mu \neq 0, \lambda \neq 0$

$$\Phi(\xi) = \ln\left(-\frac{2[\lambda(\xi + E) + 2]}{\lambda^2(\xi + E)} \right).$$

(v) When $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0$

$$\Phi(\xi) = \ln(\xi + E).$$

Step-3: We substitute (3) into (2) and use (4) so that we get a polynomial of exp(–Φ(ξ)) and from this polynomial we equate all the coefficients of same power of exp(–Φ(ξ))to zero. This procedure yields a system of algebraic equations from which we can find the constants $a_m, a_{m-1}, ... a_0$ and ω.

Step-4: Obtaining the constants from **Step-3**, and using the solutions of (4), we can construct the required travelling wave solutions of the nonlinear equation (1).

Results and Discussions

Application of the exp(–Φ(ξ))-expansion method

The system of the shallow water wave equation (Dolapcia *et al.*, 2013) is:

$$\begin{aligned} &u_t + (uv)_x + v_{xxx} = 0, \\ &v_t + u_x + vv_x = 0. \end{aligned} \tag{5}$$

The transformation $u(x,t) = u(\xi),\ \xi = x + \omega t$ reduces the system of (5) to

$$\begin{aligned} &\omega u' + v u' + u v' + v^{(3)} = 0, \\ &\omega v' + u' + v v' = 0. \end{aligned} \tag{6}$$

Integrating once the second equation of (6), we get

$$u = -\left(\omega v + \frac{v^2}{2}\right) + C, \tag{7}$$

where C is an integrating constant.

Substituting (7) into the first equation of (6), we obtain

$$2v^{(3)}(\xi) - \left(6\omega v(\xi) + 3v(\xi)^2 + 2\omega^2 - 2C\right)v'(\xi) = 0 \tag{8}$$

Let us consider that (8) has the solution of the form

$$v(\xi) = \sum_{i=0}^{m} a_i (\exp(-\Phi(\xi)))^i \qquad (a_i = const.) \tag{9}$$

where $\Phi = \Phi(\xi)$ satisfies the equation

$$\Phi'(\xi) = \exp(-\Phi(\xi)) + \mu \exp(\Phi(\xi)) + \lambda \tag{10}$$

Balancing the order of $v^{(3)}$ and $v^2 v'$ in (8), we have $m = 1$. So (9) takes the following form:

$$v(\xi) = a_1 \exp(-\Phi(\xi)) + a_0, \quad a_1 \neq 0 \tag{11}$$

where a_1, a_0 are constants to be determined later.

Substituting (11) into (8) and collecting the coefficients of $\exp(-\Phi(\xi))$, we obtain a system of algebraic equations for a_i and ω.

$$e^{-4\Phi(\xi)}: \quad 3a_1^2 - 12 = 0$$

$$e^{-3\Phi(\xi)}: \quad -6a_1a_0 - 3a_1^2\lambda - 6\omega a_1 + 24\lambda = 0$$

$$e^{-2\Phi(\xi)}: \quad -6\omega a_1\lambda - 6a_1\lambda a_0 - 6\omega a_0 - 3a_1^2\mu + 16\mu - 2\omega^2 + 14\lambda^2 - 3a_0^2 + 2C = 0$$

$$e^{-\Phi(\xi)}: \quad 16\mu\lambda - 6\omega a_0\lambda - 6\omega a_1\mu - 6a_1a_0\mu - 3a_0^2\lambda + 2\lambda^3 - 2\omega^2\lambda + 2C\lambda = 0$$

$$e^{0} \quad : \quad 4\mu^2 + 2\lambda^2\mu - 2\omega^2\mu - 3a_0^2\mu - 6\omega a_0\mu + 2C\mu = 0$$

Solving the algebraic equations above, yields:

$$Set\,A: \quad a_1 = 2, a_0 = \lambda \pm \sqrt{\lambda^2 - 4\mu - 2C}, \omega = \pm\sqrt{\lambda^2 - 4\mu - 2C} \tag{12}$$

$$Set\,B: \quad a_1 = -2, a_0 = -\lambda \pm \sqrt{\lambda^2 - 4\mu - 2C}, \omega = \pm\sqrt{\lambda^2 - 4\mu - 2C} \tag{13}$$

Substituting (12) and (13) into (11), we get two set of solutions for the equation (8)

$$v_A(\xi) = 2\exp(-\Phi(\xi)) + \lambda \pm \sqrt{\lambda^2 - 4\mu - 2C}$$

$$v_B(\xi) = -2\exp(-\Phi(\xi)) - \lambda \pm \sqrt{\lambda^2 - 4\mu - 2C}, \text{ where } \xi = x \pm \sqrt{\lambda^2 - 4\mu - 2C}\,t$$

Using these above solutions for $v(\xi)$ in (7), we can find $u(\xi)$ easily. And finally, we use the solutions of (10), i.e. $\Phi(\xi)$ to obtain the required solutions $u(x,t)$ and $v(x,t)$ of the system of equations (5). Solutions of (5) for various cases are discussed below:

Case-1: When $\lambda^2 - 4\mu > 0, \mu \neq 0$

$$v_{A,1}(x,t) = \lambda \pm \sqrt{\lambda^2 - 4\mu - 2C} - \frac{4\mu}{\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(x \pm \sqrt{\lambda^2 - 4\mu - 2C}\ t + E)\right) + \lambda}$$

$u_{A,1}(x,t)$

$$= -\sqrt{\lambda^2 - 4\mu - 2C}\left(\lambda \pm \sqrt{\lambda^2 - 4\mu - 2C} - \frac{4\mu}{\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(x \pm \sqrt{\lambda^2 - 4\mu - 2C}\ t + E)\right) + \lambda}\right)$$

$$-\frac{1}{2}\left(\lambda \pm \sqrt{\lambda^2 - 4\mu - 2C} - \frac{4\mu}{\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(x \pm \sqrt{\lambda^2 - 4\mu - 2C}\, t + E)\right) + \lambda}\right)^2 + C$$

and

$$v_{B,1}(x,t) = -\lambda \pm \sqrt{\lambda^2 - 4\mu - 2C} + \frac{4\mu}{\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(x \pm \sqrt{\lambda^2 - 4\mu - 2C}\ t + E)\right) + \lambda}$$

$u_{B,1}(x,t)$

$$= -\sqrt{\lambda^2 - 4\mu - 2C}\left(-\lambda \pm \sqrt{\lambda^2 - 4\mu - 2C} + \frac{4\mu}{\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(x \pm \sqrt{\lambda^2 - 4\mu - 2C}\ t + E)\right) + \lambda}\right)$$

$$-\frac{1}{2}\left(-\lambda \pm \sqrt{\lambda^2 - 4\mu - 2C} + \frac{4\mu}{\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(x \pm \sqrt{\lambda^2 - 4\mu - 2C}\ t + E)\right) + \lambda}\right)^2 + C$$

Case-2: When $\lambda^2 - 4\mu < 0, \mu \neq 0$

$$v_{A,2}(x,t) = \lambda \pm \sqrt{\lambda^2 - 4\mu - 2C} + \frac{4\mu}{\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(x \pm \sqrt{\lambda^2 - 4\mu - 2C}\ t + E)\right) - \lambda}$$

$u_{A,2}(x,t)$

$$= -\sqrt{\lambda^2 - 4\mu - 2C}\left(\lambda \pm \sqrt{\lambda^2 - 4\mu - 2C} + \frac{4\mu}{\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(x \pm \sqrt{\lambda^2 - 4\mu - 2C}\ t + E)\right) - \lambda}\right)$$

$$-\frac{1}{2}\left(\lambda \pm \sqrt{\lambda^2 - 4\mu - 2C} + \frac{4\mu}{\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(x \pm \sqrt{\lambda^2 - 4\mu - 2C}\ t + E)\right) - \lambda}\right)^2 + C$$

and

$$v_{B,2}(x,t) = -\lambda \pm \sqrt{\lambda^2 - 4\mu - 2C} - \frac{4\mu}{\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(x \pm \sqrt{\lambda^2 - 4\mu - 2C}\ t + E)\right) - \lambda}$$

$$u_{B,2}(x,t)$$

$$= -\sqrt{\lambda^2-4\mu-2C}\left(-\lambda \pm \sqrt{\lambda^2-4\mu-2C} - \frac{4\mu}{\sqrt{4\mu-\lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(x\pm\sqrt{\lambda^2-4\mu-2C}\ t+E)\right)-\lambda}\right)$$

$$-\frac{1}{2}\left(-\lambda \pm \sqrt{\lambda^2-4\mu-2C} - \frac{4\mu}{\sqrt{4\mu-\lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(x\pm\sqrt{\lambda^2-4\mu-2C}\ t+E)\right)-\lambda}\right)^2 + C$$

Case-3: When $\lambda^2-4\mu>0, \mu=0, \lambda\neq 0$

$$v_{A,3}(x,t) = \lambda \pm \sqrt{\lambda^2-2C} + \frac{2\lambda}{e^{\lambda\left(x\pm\sqrt{\lambda^2-2C}\,t+E\right)}-1}$$

$$u_{A,3}(x,t) =$$

$$-\sqrt{\lambda^2-2C}\left(\lambda \pm \sqrt{\lambda^2-2C} + \frac{2\lambda}{e^{\lambda\left(x\pm\sqrt{\lambda^2-2C}\,t+E\right)}-1}\right)$$

$$-\frac{1}{2}\left(\lambda \pm \sqrt{\lambda^2-2C} + \frac{2\lambda}{e^{\lambda\left(x\pm\sqrt{\lambda^2-2C}\,t+E\right)}-1}\right)^2 + C$$

and

$$v_{B,3}(x,t) = -\lambda \pm \sqrt{\lambda^2-2C} - \frac{2\lambda}{e^{\lambda\left(x\pm\sqrt{\lambda^2-2C}\,t+E\right)}-1}$$

$$u_{B,3}(x,t) = -\sqrt{\lambda^2-2C}\left(-\lambda \pm \sqrt{\lambda^2-2C} - \frac{2\lambda}{e^{\lambda\left(x\pm\sqrt{\lambda^2-2C}\,t+E\right)}-1}\right)$$

$$-\frac{1}{2}\left(-\lambda \pm \sqrt{\lambda^2-2C} - \frac{2\lambda}{e^{\lambda\left(x\pm\sqrt{\lambda^2-2C}\,t+E\right)}-1}\right)^2 + C$$

Case-4: When $\lambda^2-4\mu=0, \mu\neq 0, \lambda\neq 0$

$$v_{A,4}(x,t) = \lambda \pm \sqrt{-2C} - \frac{\lambda^2\left(x\pm\sqrt{-2C}\,t+E\right)}{\lambda(x\pm\sqrt{-2C}\,t+E)+2}$$

$$u_{A,4}(x,t) = -\sqrt{-2C}\left(\lambda \pm \sqrt{-2C} - \frac{\lambda^2(x \pm \sqrt{-2C}\,t + E)}{\lambda(x \pm \sqrt{-2C}\,t + E) + 2}\right)$$
$$-\frac{1}{2}\left(\lambda \pm \sqrt{-2C} - \frac{\lambda^2(x \pm \sqrt{-2C}\,t + E)}{\lambda(x \pm \sqrt{-2C}\,t + E) + 2}\right)^2 + C$$

and

$$v_{B,4}(x,t) = -\lambda \pm \sqrt{-2C} + \frac{\lambda^2\left(x \pm \sqrt{-2C}\,t + E\right)}{\lambda(x \pm \sqrt{-2C}\,t + E) + 2}$$

$$u_{B,4}(x,t) = -\sqrt{-2C}\left(-\lambda \pm \sqrt{-2C} + \frac{\lambda^2(x \pm \sqrt{-2C}\,t + E)}{\lambda(x \pm \sqrt{-2C}\,t + E) + 2}\right)$$
$$-\frac{1}{2}\left(-\lambda \pm \sqrt{-2C} + \frac{\lambda^2(x \pm \sqrt{-2C}\,t + E)}{\lambda(x \pm \sqrt{-2C}\,t + E) + 2}\right)^2 + C$$

Case-5: When $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0$

$$v_{A,5}(x,t) = \pm\sqrt{-2C} + \frac{1}{x \pm \sqrt{-2C}\,t + E}$$

$$u_{A,5}(x,t) = -\sqrt{-2C}\left(\pm\sqrt{-2C} + \frac{1}{x \pm \sqrt{-2C}\,t + E}\right)$$
$$-\frac{1}{2}\left(\pm\sqrt{-2C} + \frac{1}{x \pm \sqrt{-2C}\,t + E}\right)^2 + C$$

and

$$v_{B,5}(x,t) = \pm\sqrt{-2C} - \frac{1}{x \pm \sqrt{-2C}\,t + E}$$

$$u_{B,5}(x,t) = -\sqrt{-2C}\left(\pm\sqrt{-2C} - \frac{1}{x \pm \sqrt{-2C}\,t + E}\right)$$
$$-\frac{1}{2}\left(\pm\sqrt{-2C} - \frac{1}{x \pm \sqrt{-2C}\,t + E}\right)^2 + C$$

Graphical representation

Here, we have sketched graphs of the solutions for the case-1 only, where we get various types of solution profiles: kink-type, singular etc. Fig.1 and Fig.2, Fig.5 and Fig.6 describe for positive values of a_0 and ω and the other four are for negative values of a_0 and ω. In the similar way, we can draw figures for the other solutions easily.

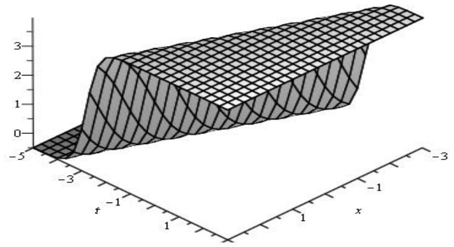

Fig. 1: Kink-type soliton profile of $v_{A,1}(x,t)$ when $\lambda=3, \mu=1, C=1, E=1$.

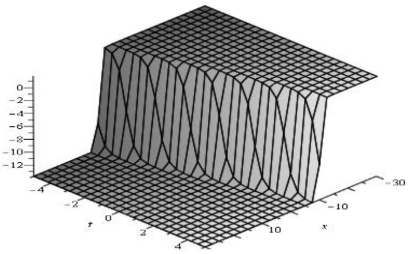

Fig.2: Kink-type soliton profile of $u_{A,1}(x,t)$ when $\lambda=3, \mu=1, C=1, E=1$.

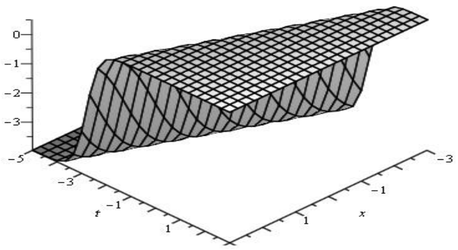

Fig. 3: Kink-type soliton profile of $v_{A,1}(x,t)$ when $\lambda=3, \mu=1, C=1, E=1$

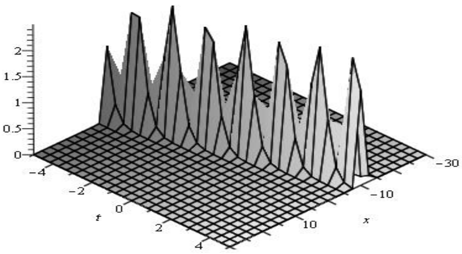

Fig. 4: Singular soliton profile of $u_{A,1}(x,t)$ when $\lambda=3, \mu=1, C=1, E=1$

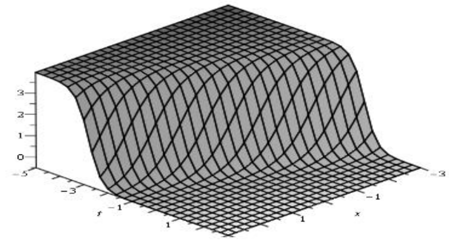

Fig. 5: Kink-type soliton profile of $v_{B,1}(x,t)$ when $\lambda=3, \mu=1, C=1, E=1$

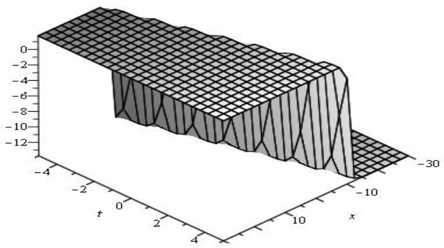

Fig. 6: Kink-type soliton profile of $u_{B,1}(x,t)$ when $\lambda=3, \mu=1, C=1, E=1$

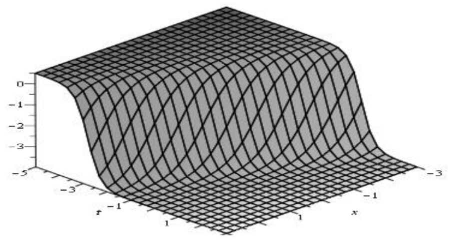

Fig. 7: Kink-type soliton profile of $v_{B,1}(x,t)$ when $\lambda=3, \mu=1, C=1, E=1$

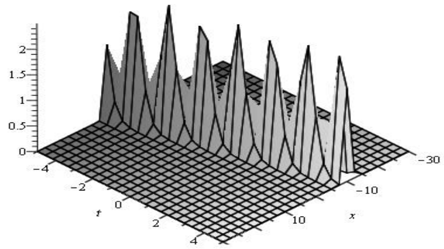

Fig. 8: Singular soliton profile of $u_{B,1}(x,t)$ when $\lambda=3, \mu=1, C=1, E=1$

Conclusion

These solutions may be important for explaining some practical physical phenomena which are modeled by this equation. It can be concluded that the $\exp(-\Phi(\xi))$-expansion method is a very powerful and efficient technique in finding exact solutions for wide classes of nonlinear problems. Another possible merit is that the reliability of the method and the reduction in the size of computational domain give this method a wider applicability.

References

Abdou, M.A., (2007). The extended tanh-method and its applications for solving nonlinear physical models. *Appl. Math. Comput*., **190**: 988-996.

Bakir, A. and A. Boz, (2008). Exact solutions for nonlinear evolution equation using Exp-function method. *Physics Letters A,* **372**: 1619-1625.

Dolapcia, I., Y. Timuçin and Ahmet, (2013). Some exact solutions to the generalized Korteweg–deVries equationa nd the system of shallow water wave equations. *Nonlinear Analysis: Modelling and Control,* **18(**1): 27-36.

He, J.H., (1999). Variational iteration method a kind of non-linear analytical technique: some examples. *Int. J. Nonlinear Mech*, **34**: 699-708.

Islam, M.E., K. Khan, M.A. Akbar and R. Islam, (2013). Traveling Wave Solutions of Nonlinear Evolution Equation via Enhanced (G'/G)-expansion Method. *GANIT J. Bangladesh Math. Soc*., **33**: 83 - 92.

Jawad, A.J.M., M.D. Petkovi´c and A. Biswas, (2010). Modified simple equation method for nonlinear evolution equations. *Applied Mathematics and Computation*, **217(**2): 869-877.

Khan, K. and M.A. Akbar, (2013). Application of the exp(–Φ(ξ))-expansion method to Find the Exact Solutions of Modified Benjamin-Bona-Mahony Equation. *World Applied Sciences Journal*, **24**(10): 1373-1377.

Salam, M.A., (2012). Traveling-Wave Solution of Modified Liouville Equation by Means of Modified Simple Equation Method. ISRN *Applied Mathematics*, Article ID 565247, 4 pages doi:10. 5402/ 2012/565247.

Wang, M., Zhou Y. and Z. Li, (1996). Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics. *Physics Letters A,* **216(**1-5): 67-75.

Zayed, E.M.E., Zedan H.A. and K.A. Gepreel, (2004). Group analysis and modified tanh-function to find the invariant solutions and soliton solution for nonlinear Euler equations. *International journal of Nonlinear Sci. Numer. Simul.,* **5**: 221-234.