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Normal Separation Axiom on Fuzzy Bitopological Space

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ABSTRACT

In our paper, we present three novel concepts related to the normal separation property within the realm of fuzzy bitopological spaces (FPTS), specifically focusing on pairwise fuzzy normal bitopological spaces (FPN). These notions are introduced in a quasi-coincidence sense, and we establish relationships between our propositions and other existing ones. Furthermore, we provide proofs demonstrating that all the introduced concepts exhibit the 'good extension' property. Notably, we observe that our notions maintain their characteristics under one-one, onto, fuzzy pairwise open (FP-open), fuzzy closed (FP-closed), and fuzzy pairwise continuous (FP-continuous) mappings.

1. Introduction

In 1965 the notion of fuzzy set was first explored by L Ali Zadeh (Zadeh, 1965). Using this notion, C L Chang introduced (Chang, 1968) fuzzy topological space (FTS). After that various works on FTS have been developed in various directions; in particular, fuzzy normality (Hutton, 1975, Miah *et al.*, 2018), fuzzy regularity (Ali, 1990a), separations on fuzzy topological spaces (Ali, 1990b, Amin *et al*, 2014, Miah & Amin, 2017a), fuzzy bitopological spaces (Hossain and Ali, 2007, Amin *et al*, 2014, Miah *et al.*, 2019a), Gouguen (1973), Wong (1974), Lowen (1976), Warren (1974), Hutton (1975) and others. Separation axioms (SA) are playing a significant role on FTS (Miah *et al.*, 2017b) and on bitopological (Miah *et al.*, 2019b) spaces. Many works (Ali, 1990b), (Ahmd, 1989), (Miah & Amin, 2017b), (Miah *et al.*, 2017a) on SA have been done by academicians and researchers around the world. Out of the axioms, FPN type is more attractive to researchers and it has been already stated in fuzzy bitopology. The works on FPN which are studied by Wuyts and Lowen (1983), Ali (1990b), Guler and Kale (2015) and distinguished researchers.

The main objective of our paper is to contribute to the extension of SA on FPN. In the present work, FPN is defined by using the sense of quasi-coincidence and implications among ours and counterpart notions are studied. It is showed that the good-extension, hereditary and homomorphism preserving features hold for the discussed concepts.

2. Fuzzy Normal Bitopological Space

Here, we have studied the new notion and results. Different well-known attributes are discussed and proved using the given concepts.

Definition: A FPTS (X,s,t) is said to be

(a) FPN(i) iff $w_1\bar{q}w_2$, w_1 is s-closed fuzzy set, w_2 is t-closed fuzzy set, $\exists\ \varrho\in N(w_1,s)$ and $\vartheta\in N(w_2,t)$ such that (s. t.) $\varrho\bar{q}\vartheta$ (Safiya *et al.*, 1994).

(b) FPN(ii) iff $w_1\bar{q}w_2$, w_1, w_2 are $s\cup t$-closed fuzzy sets, $\exists\ \varrho\in N(w_1,s\cup t)$ and $\vartheta\in N(w_2,s\cup t)$ s. t. $\varrho\cap\vartheta=0$.

(c) FPN(iii) iff $w_1\cap w_2=0$, w_1, w_2 are $s\cup t$-closed fuzzy sets, $\exists\ \varrho,\vartheta\in s\cup t$ s. t. $w_1\subseteq\varrho$, $w_2\subseteq\vartheta$ and $\varrho\bar{q}\vartheta$.

(d) FPN(iv) iff $w_1\cap w_2=0$, w_1, w_2 are $s\cup t$-closed fuzzy sets, $\exists\ \varrho,\vartheta\in s\cup t$ s. t. $w_1\subseteq\varrho$, $w_2\subseteq\vartheta$ and $\varrho\cap\vartheta=0$.

Theorem: If (X,s,t) if a FPTS then the implications (b)⇒(a), (d)⇒(c) hold. But the reverses are not true in general.

Proof: (b)⇒(a), (d)⇒(c) are obvious since $\varrho\cap\vartheta=0$ implies that $u\bar{q}\vartheta$.

To show (a)⇏(b) there is the following example.

Proof: (b)⇒(a), (d)⇒(c) are obvious since $\varrho\cap\vartheta=0$ implies that $u\bar{q}\vartheta$.

To show (a)⇏(b) there is the following example.

Example: If X={x,y} and s be a fuzzy topology on X created by {ϱ, ϑ} and t be a topology on X generated by {constants fuzzy sets}, where $\varrho(x)=0.4$, $\varrho(y)=0.2$ and $\vartheta(x)=0.3$, $\vartheta(y)=0.6$. Then we see that (X, s, t) is FPN(i) but not FPN(ii). Hence (a) ⇏(b).

To show (c)⇏(d) there is the following example.

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Example: If $X=\{x, y\}$ and s be a fuzzy topology on X created by $\{\varrho, \vartheta\}$ and t be a topology on X generated by {constants fuzzy sets},where $\varrho(x)=0.4$, $\varrho(y)=0.2$ and $\vartheta(x)=1$, $\vartheta(y)=0.3$. Then we see that (X, s, t) is FPN(iii) but not FPN(iv). Hence, (c) $\not\Rightarrow$(d).

To show the good extension characteristic of the defined notions the following theorem is sufficient.

Theorem: The bitopological space (X, S, T) is pairwise normal if and only if (X, w(S), w(T)) is FPN(j), where j=i, ii, iii, iv.

Proof: Here we will prove the theorem for j=iv and similarly one can prove the others. Suppose (X, w(S), w(T)) is FPN(iv). We have to show that (X, S, T) is pairwise normal. Let W_1, W_2 be S ∪ T-closed sets with $W_1 \cap W_2=\emptyset$. Then it is obvious that 1_{W1} and 1_{W2} are fuzzy sets in w(S)∪w(T) respectively and $1_{W1\cap W2}= 0$. But $1_{W1\cap W2}= 0$ implies

$1_{W1} \cap 1_{W2}=0$. Since (X, w(S), w(T)) is FPN(iv), then $\exists\ \varrho, \vartheta \in$(w(S)∪w(T)) s. t. $1_{W1} \subseteq \vartheta$, $1_{W2} \subseteq \varrho$ and $\varrho \cap \vartheta=0$.

Since $\varrho, \vartheta \in$(w(S)∪w(T)), then $\varrho^{(-1)}$ (0,1], $\vartheta^{(-1)}$(0,1]∈S∪T. Again since $1_{W1} \subseteq \vartheta$, then $\vartheta^{(-1)}(0,1] \supseteq (1_{W1})^{(-1)}$ (0,1]=W_1. Similarly, $W_2 \subseteq \varrho^{(-1)}$ (0,1].

Suppose that $\varrho^{(-1)}$ (0,1]$\cap \vartheta^{(-1)}$ (0,1]$\neq \phi$, then $\exists$ x∈X s. t. $\varrho(x) > 0$ and $\vartheta(x) > 0$. So $(\varrho \cap \vartheta)(x) > 0$ contradicts with $\varrho \cap \vartheta=0$.

Hence (X, S, T) is pairwise normal bitopological space.

Conversely, let the bitopological space (X, S, T) be pairwise normal. We will show that (X, w(S), w(T)) is FPN(iv). Let w_1, w_2 are w(S)∪w(T)-closed with $w_1 \cap w_2=0$. Then w_1^c, $w_2^c \in$(w(S)∪w(T)) and $(w_1^c)^{(-1)}$ (0,1], $(w_2^c)^{(-1)}$ (0,1]∈S∪T. Also,

$((w_1^c)^{(-1)} (0,1])^c=w_1^{(-1)}\{1\}$ and $((w_2^c)^{(-1)} (0,1])^c = w_2^{(-1)}\{1\}$. It is clear that $w_1^{(-1)}\{1\}$ and $w_2^{(-1)}\{1\}$ are closed sets in ∪T. Since (X, S, T) is pairwise normal, then ∃ U, V∈S∪T s. t.

$w_1^{(-1)}\{1\} \subseteq V$, $w_2^{(-1)}$ {1⊆U and $U \cap V=\emptyset$.

Thus, it is clear that 1_V, $1_U \in$w(S)∪w(T) and $1_U \cap 1_V= 0$ as $1_{(U\cap V)}=0$ implies

$1_U \cap 1_V=0$. Hence $1_{(w1(-1)\{1\})}= w_1 \subseteq 1_V$, $1_{(w2(-1)\{1\})}= w_2 \subseteq 1_U$ and $1_U \cap 1_V=0$.

Hence (X, w(S), w(T)) is FPN(iv).

To show the hereditary characteristic of the defined notions the following theorem is sufficient.

Theorem: If (X, s, t) be a fuzzy bitopological space and E⊆X, $s_E=\{\varrho/E:\varrho \in s\}$, $t_E=\{\vartheta/E:\vartheta \in t\}$ and (X, s, t) is FPN(j) then (E, s_E, t_E) is FPN(j), where j=i, ii, iii, iv.

Proof: The proof is obvious.

Theorem: Let (X, s, t) and (Y, s_1, t_1) be two FPTSs and f:X⟶Y be bijective, FP-continuous and FP-open. Then (X, s, t) is FPN(j)⇒(Y, s_1, t_1) is FPN(j), where j=i, ii, iii, iv.

Proof: Let us consider the FPTS (X, s, t) is FPN(iii). We will show that (Y, s_1, t_1) is FPN(iii). Let w_1, w_2 are $s_1 \cup t_1$-closed with $w_1 \cap w_2=0$. Then $f^{(-1)}(w_1)$, $f^{(-1)}(w_2) \in s^c \cup t^c$ as f is FP-continuous. Let x∈X then, we have f(x)=y for y in Y. Now, for all x∈X, we have

$(f^{(-1)}(w_1) \cap f^{(-1)}(w_2))(x)=w_1 (f(x)) \cap w_2 (f(x))=w_1 (y) \cap w_2 (y)=0$, as $w_1 \cap w_2=0$.

Since (X, s, t) is FPN(iii), ∃ $\varrho, \vartheta \in$s∪t such that $f^{(-1)}(w_1) \subseteq \varrho$, $f^{(-1)}(w_2) \subseteq \vartheta$ and $\varrho \bar{q} \vartheta$.

Now, $f^{(-1)}(w_1) \subseteq \varrho$, $f^{(-1)}(w_2) \subseteq \vartheta$ imply $w_1 \subseteq (\varrho)$, $w_2 \subseteq f(\vartheta)$ as f is bijective.

Also $\varrho \bar{q} \vartheta \Rightarrow \varrho(x)+\vartheta(x) \leq 1, \forall\ x \in X$.

Again, for all f(x)∈Y, we have $f(\varrho)\ f(x) + f(\vartheta)(f(x))=\varrho(x) + \vartheta(x) \leq 1$ as f is bijective. So that $f(\varrho) \bar{q} f(\vartheta)$.

Now, it is clear that $f(\varrho)$, $f(\vartheta) \in s_1 \cup t_1$ as f is FP-open. So, $f(\varrho)$, $f(\vartheta) \in s \cup t_1$ such that $w_1 \subseteq f(\varrho)$, $w_2 \subseteq f(\vartheta)$ and $f(\varrho)\ \bar{q}\ f(\vartheta)$. Hence (Y, s_1, t_1) is FPN(iii).

The proofs for j=i, ii, iv are of similar manners.

Theorem: Let (X, s, t) and (Y, s_1, t_1) be two FPTSs and f:X⟶Y be bijective, FP-continuous and FP-closed. Then (Y, s_1, t_1) is FPN(j)⇒(X, s, t) is FPN(j), where j=i, ii, iii, iv.

Proof: Suppose the FPTS (Y, s_1, t_1) is FPN(iii). We will show that (X, s, t) is FPN(iii). Let w_1, w_2 are s ∪ t-closed with $w_1 \cap w_2=0$. Then $f(w_1)$, $f(w_2) \in s_1^c \cup t_1^c$ as f is FP-closed. Now for all y∈Y, we have

$(f(w_1) \cap f(w_2))(y)=(f(w_1) \cap f(w_2))(f(x))=f(w_1)(f(x)) \cap f(w_2)(f(x))$

$= w_1(x) \cap w_2(x)=0$,

since y=f(x) and f is bijective.

Since (Y, s_1, t_1) is FPN(iii), then ∃ $\varrho, \vartheta \in s_1 \cup t_1$ such that $f(w_1) \subseteq \varrho$, $f(w_2) \subseteq \vartheta$ and $\varrho \bar{q} \vartheta$.

Now, we see that $f(w_1) \subseteq \varrho$, $f(w_2) \subseteq \vartheta$ imply that $w_1 \subseteq f^{(-1)}(\varrho)$, $w_2 \subseteq f^{(-1)}(\vartheta)$ as f is bijective.

Again $u\bar{q}v$ implies that $\varrho(y)+\vartheta(y) \leq 1$ for all y∈Y.

Now for all x∈X, we have $f^{(-1)}(\varrho)(x)+f^{(-1)}(\vartheta)(x)=\varrho(f(x))+\vartheta(f(x))=\varrho(y)+\vartheta(y) \leq 1$. So $f^{(-1)}(\varrho)\ \bar{q}\ f^{(-1)}(\vartheta)$.

Since f is FP-continuous, then $f^{(-1)}(\varrho)$, $f^{(-1)}(\vartheta) \in$s∪t. So, we get $f^{(-1)}(\varrho)$, $f^{(-1)}(\vartheta) \in$s∪ t such that $w_1 \subseteq f^{(-1)}(\varrho)$, $w_2 \subseteq f^{(-1)}(\vartheta)$ and $f^{(-1)}(\varrho)\bar{q}f^{(-1)}(\vartheta)$. Hence (X, s, t) is FPN(iii).

Similarly, one can prove the others for j=i, ii, iv.

3. Conclusion

Separation is an important area of fuzzy bitopology in fuzzy mathematics. In this research, we have studied and discussed some definitions of normal-type separation in fuzzy bitopological space in the approach of quasi-coincidence. We have proved all the definitions are good extension of the existing counterparts and FPN(ii) is finer than other such definition (Safiya *et al*, 1994). We expect that all outputs of this article will be supportive for investigators to bring out a general outline for more development of separation axiom (normal) on fuzzy bitopologies.

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Appendix A

Basic notions and Preliminary Results Here, we recall some concepts those are presented in the cited papers which will be needed in the sequel. In this paper, X and Y are always denoted as non-empty sets and I=[0,1]. The class of all fuzzy sets on a non-empty set X is denoted by I^X and fuzzy sets on X are denoted by u, v, w etc.

A.1. Definition

A function u from X into the unit interval I is called a fuzzy set in X. For every x∈X, u(x)∈I is called the grade of membership of x in u. Some authors say u is a fuzzy subset X. Thus a usual subset of X, is a special type of a fuzzy set in which the ranges of the function is {0, 1}. The class of all fuzzy sets from X into the closed unit interval I will be denoted by I^X (Zadeh, 1965).

A.2. Definition

A fuzzy set uin X is called a fuzzy singleton if and only if u(x) = r, 0 < r ≤ 1, for a certain x∈Xandu(y) = 0 for all points y of X except x. The fuzzy singleton is denoted by x_r and x is its support. The class of all fuzzy singletons in X will be denoted by S(X). If $u \in I^X$ and $x_r \in S(X)$, then we say that $x_r \in u$ if and only if r ≤ u(x) (Ming and Ming, 1980a).

A.3. Definition

A fuzzy singleton x_r is said to be quasi-coincidence with u, denoted by x_r qu if and only if $u(x) + r > 1$. If x_r is not quasi-coincidence with u, we write $x_r \bar{q}$ u and defined as $u(x) + r \leq 1$ (Kandil and El-Shafee, 1991).

A.4. Definition

Let f be a mapping from a set X into a set Y and u be a fuzzy subset of X. Then f and u induce a fuzzy subset v of Y defined by
$v(y)=\sup\{u(x)\}$ if $x \in f^{(-1)} [\{y\}] \neq \varphi$, $x \in X$ $= 0$ otherwise (Chang, 1968).

A.5. Definition

Let f be a mapping from a set X into a set Y and v bea fuzzy subset of Y . Then the inverse of v written as $f^{(-1)}$ (v) is a fuzzy subset of X defined by $f^{(-1)}(v)(x) = v(f(x))$, for $x \in X$ (Chang, 1968).

A.6. Definition

Let I = [0, 1], X be a non-empty set and I^X be the collection of all mappings from X into I, i.e. the class of all fuzzy sets in X. A fuzzy topology on X is defined as a family t of members of I^X, satisfying the following conditions.
(i) $1, 0 \in t$,
(ii) If $u \in t$ for each $i \in \Lambda$, then $\cup_{(i \in \Lambda)} u_i \in t$, where Λ is an index set.
(iii) If u, $v \in t$ then $u \cap v \in t$(Chang, 1968).
The pair (X, t) is called a fuzzy topological space (in short fts) and members of t are called t- open fuzzy sets. A fuzzy set v is called a t-closed fuzzy set if $1-v \in t$.

A.7. Definition

The function $f:(X, t) \rightarrow (Y, s)$ is called fuzzy continuous if and only if for every $v \in s$, $f^{(-1)}(v) \in t$, the function f is called fuzzy homeomorphic if and only if f is bijective and both f and $f^{(-1)}$ are fuzzy continuous (Ming and Ming, 1980b).

A.8. Definition

The function $f:(X, t) \rightarrow (Y, s)$ is called fuzzy open if and only if for every open fuzzy set u in (X, t), f(u) is open fuzzy set in (Y, s) (Malghan and Benchalli, 1984).

A.9. Definition

The function $f:(X, t) \rightarrow (Y, s)$ is called fuzzy closed if and only if for every closed fuzzy set u in (X, t), f(u) is closed fuzzy set in (Y, s) (Ming and Ming, 1980b).

A.10. Definition

A fuzzy bitopological space (fbts, in short) is a triple (X, s, t) where s and t are arbitrary fuzzy topologies on X (Kandil *et al*., 1995).

A.11. Definition

A bitopological space (X, S, T) is called pairwise normal (in short, PN) if, given S-closed set A and a T-closed set B with $A \cap B = \emptyset$, there exist $U \in S$, $V \in T$ such that $A \subseteq V$, $B \subseteq U$ and $U \cap V = \emptyset$ (Kelly, 1963).

A.12. Definition

A bijective mapping from an fts (X, t) to an fts (Y, s) preserves the value of a fuzzy singleton (fuzzy point).

Note: Preimage of any fuzzy singleton (fuzzy point) under bijective mapping preserves its value (Amin *et al*., 2014).