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Numerical Solution of Composite Fractional Oscillation Equation by Fractional Differential Transform Method and Variational Iteration Method

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ABSTRACT

This research aims to find the numerical solution of nonlinear composite fractional oscillation equation using Fractional Differential Transform Method (FDTM) and Variational Iteration Method (VIM). The numerical solutions demonstrate that the two approaches agree fairly well. As a result, these two approaches represent extremely strong and effective methods for resolving various types of fractional differential equations, both linear and non-linear, that arise in numerous technological and engineering domains. We can conclude that both approaches are highly effective and potent in obtaining both numerical and analytical resolutions for a broad range of FDEs. However, the FDTM solves nonlinear problem more closer to the exact solution, giving it an advantage over the VIM.

1. Introduction

In applied mathematics, integrals and derivatives of random directives are treated using fractional calculus methods. Fractional calculus has been used in a wide range of seemingly unrelated scientific and engineering domains over the past ten years. Equations in fluid dynamics, Sound and hearing, biology, electromagnetics, dispersion, signal handling, and many other physical mechanisms are increasingly being modeled using fractional differential equations (Podlubny, 1999).

These operations in calculus are extended to fractal orders in calculating fractions. Because fractional order operators are non-local and capture the dynamics' history, Fractional Calculus has turned into a crucial element for the study of dynamical systems.

It is possible to characterize the fluctuating systems of complicated entities non-local processes with storage using a fractional equations of motion. The relationship between stochastic differential equation and nonlocal differential equations can be established using it.

It is discovered that fractional calculus is better suited for simulating processes using a long-term interaction as well as concrete issues expressed by fractal equations; yet, solving fractional differential equations can occasionally be challenging.

Because of this, we require a dependable and effective technique in resolving Differential calculus with fractions. A fractional time Klein-Gordon formula analytical research is provided by (Tamsir *et al.*, 2010). Chen *et al.* (2017) analyze the fractional time Klein-Gordon formula applying the discrete technique. Researchers are less likely to offer an approximation of the answer. We provide an analytical solution for the temporal fractional difference in this study.

The Riemann-Liouville fractal derivative, the Grünwald-Letnikov fractal derivative, the Rietz fractal derivative, also many more variants exist. When it comes to genuine mathematics, the Riemann-Liouville integral is a little greater well-liked comparing the Caputo derivatives. The Riemann-Liouville derivatives require us that we define specify the numerical values of a few fractional derivatives on the starting conditions for the unidentified answer, yet it was used by many earlier scholars in place of the Caputo derivative. Nevertheless, the fractional derivative has little physical significance when we tackle the actual physical issue. Only the integer order derivative may be specified when working with the Caputo derivatives. It is measurable and has an obvious physical purpose. The fact that the equations containing the Riemann-Liouville operator are identical under homogeneous conditions is

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another reason we use the Caputo derivative.

A substantial body of literature has recently emerged on the use of fractal differential equations in non-linear mechanics (Gao & Yu, 2005; Lu & Chen, 2005, 2006). Since there are rarely exact data-driven solutions for fractal differential equations, approximation also numerical approaches are required. Two relatively recent techniques for providing an data-driven estimation to linear and nonlinear problems are VIM (He, 1997, 1998a, 1999, 2000,) and the Adomian decomposition method (Adomian, 1998). These techniques are especially useful as elements for researcher and applied scientists (Nomani & Al-Khaled, 2005).

Both of them offer data driven estimations to take linear and non-linear difference equations the need for separate or regression, as well as instantaneous as well as obvious metaphorical expressions of data-derived solutions. The two approaches are used in this study to find analytical approximation solutions to fractional-order linear differential equations. Afterwards, several examples are provided to illustrate an analysis based on numbers with the fractal difference. many different linear or non-linear difference equations have approximate solutions found in the publications thanks to the ADM method (Shawagfeh & Kaya, 2004). The technique's application to FDEs has been lately expanded (Nomani, 2005).

While the Laplacian transform technique can solve some constant coefficient fractional differential equations, it requires forcing terms, so it is not suited to all fractal inequality equations with coefficients that are fixed.

For the purpose of finding the solitary solutions to non-linear difference equations, non-linear differential-difference equations, and nonlinear fractal differential problems, a very thorough analysis of research has been conducted recently (Tamsir and Srivastava, 2016).

2. Methods

2.1. Non-Homogeneous Two Terms Fractional Differential Equations Involving Caputo Fractional Derivative:

Let us introduce a non-homogeneous fractional differential equation of the form:

$$ {}_0D_x^{\alpha} y(x) + {}_0D_x^{\beta} y(x) = h(x) \quad x > 0 \ , \ 0 < \alpha < \beta < 1 \ , \ y(0) = c $$

and $L\{{}_0D_x^{\alpha} y(x)\} + L\{{}_0D_x^{\beta} y(x)\} = L\{h(x)\}$

and

Then

$$ s^{\alpha}Y(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} y^{(k)}(0) + s^{\beta}Y(s) - \sum_{k=0}^{n-1} s^{\beta-k-1} y^{(k)}(0) = H(s) \quad \text{where} \quad n-1 < \alpha \le n $$

Now, when, $n = 1$

$$ s^{\alpha}Y(s) - s^{\alpha-1}y(0) + s^{\beta}Y(s) - s^{\beta-1}y(0) = H(s) $$

$$ \Rightarrow (s^{\alpha} + s^{\beta})Y(s) - (s^{\alpha-1} + s^{\beta-1})y(0) = H(s) $$

$$ \Rightarrow (s^{\alpha} + s^{\beta})Y(s) - (s^{\alpha-1} + s^{\beta-1})c = H(s) $$

$$ \Rightarrow (s^{\alpha} + s^{\beta})Y(s) = H(s) + (s^{\alpha-1} + s^{\beta-1})c $$

$$ \Rightarrow Y(s) = \frac{H(s)}{s^{\alpha} + s^{\beta}} + \frac{s^{\alpha-1} + s^{\beta-1}}{s^{\alpha} + s^{\beta}} c $$

$$ \Rightarrow Y(s) = H(s) \cdot \frac{1}{s^{\alpha} + s^{\beta}} + \frac{1}{s} c $$

$$ y(x) = L^{-1}\{G(s) \cdot H(s)\} + c.(1) $$

Where, $G(s) = \frac{1}{s^{\alpha} + s^{\beta}} = \frac{1}{s^{\alpha}\left(1 + \frac{s^{\beta}}{s^{\alpha}}\right)} = \frac{s^{-\alpha}}{1 + s^{\beta-\alpha}}$;

$$ g(x) = L^{-1}\left\{\frac{s^{-\alpha}}{s^{\beta-\alpha-1}}\right\} = L^{-1}\left\{\frac{s^{\alpha-\beta}}{s^{\alpha} - \lambda}\right\} = x^{\beta-1}E_{\alpha,\beta}(\lambda . x^{\alpha}) $$

Comparing, $\alpha - \beta = -\alpha$; $\alpha = \beta - \alpha$; $\lambda = -1$

$$ g(x) = x^{\beta-1}E_{\beta-\alpha,\beta}(-1, x^{\beta-\alpha}); $$

$$ y(x) = L^{-1}\{G(s) \cdot H(s)\} + c $$

Convolution of two functions f and g

$$ f * g = \int_0^x f(x-\tau)g(\tau)d\tau \ ; \ L(f * g) = F(s) \cdot G(s); $$

$$ f * g = L^{-1}\{F(s) \cdot G(s)\}; \ y(x) = g(x) * h(x) + c \ ; $$

$$ y(x) = \int_0^x g(x-\tau)h(\tau)d\tau + c; $$

$$ \therefore y(x) = \int_0^x (x-\tau)^{\beta-1} E_{\beta-\alpha,\alpha}\left(-1 \cdot (x-\tau)^{\beta-\alpha}\right)h(\tau)d\tau + c $$

2.2. Non-Homogeneous Fractional Differential Equations Involving Caputo Fractional Derivative:

Now let's introduce the following non-homogeneous fractional differential equation:

$$ {}_0D_x^{\alpha} y(x) + \lambda y(x) = h(x) \ , \quad x > 0, \ n-1 < \alpha < n, $$

$$ y^k(0) \ , \ k = 0,1,2,3,.......,n-1 $$

Using Laplace Transform we get,

$$ L\{{}_0D_x^{\alpha} y(x)\} + \lambda L\{y(x)\} = L\{h(x)\} $$

$$ \Rightarrow s^{\alpha}Y(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} y^{(k)}(0) + \lambda Y(s) = H(s) $$

$$ \Rightarrow (s^{\alpha} + \lambda)Y(s) = H(s) + \sum_{k=0}^{n-1} s^{\alpha-k-1} \cdot b_k $$

$$ \Rightarrow Y(s) = \frac{H(s)}{s^{\alpha} + \lambda} + \frac{\sum_{k=0}^{n-1} s^{\alpha-k-1} \cdot b_k}{s^{\alpha} + \lambda} $$

$$ \Rightarrow y(x) = L^{-1}\{H(s) \cdot G(s)\} + \sum_{k=0}^{n-1} b_k \cdot L^{-1}\left\{\frac{s^{\alpha-k-1}}{s^{\alpha} + \lambda}\right\} $$

Using the Laplace Transform's characteristics, we can get

$$ G(s) = \frac{1}{s^{\alpha} + \lambda} = \frac{s^0}{s^{\alpha} + \lambda}; \quad L^{-1}\left\{\frac{s^{\alpha-\beta}}{s^{\alpha} - \lambda}\right\} = x^{\beta-1}E_{\alpha,\beta}(\lambda x^{\alpha}) $$

Now, we have

$\alpha - \beta = 0 \Rightarrow \alpha = \beta$

$\lambda = -\lambda$

$$g(x) = L^{-1}\left\{\frac{1}{s^{\alpha}+\lambda}\right\}$$
$$= x^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda x^{\alpha}\right)$$

The Caputo fractional derivative definition gives us

$$L^{-1}\left\{\frac{s^{\alpha-k-1}}{s^{\alpha}+\lambda}\right\} = x^{1+k-1}E_{\alpha,k+1}\left(-\lambda x^{\alpha}\right)$$
$$= x^{k}E_{\alpha,k=1}\left(-\lambda x^{\alpha}\right)$$

Then, we get

$$y(x) = g(x) * h(x) + \sum_{k=0}^{n-1} b_k \cdot x^k \cdot E_{\alpha,k+1}\left(-\lambda x^{\alpha}\right)$$

$$y(x) = \int_0^x g(x-t)h(t)dt + \sum_{k=0}^{n-1} E_{\alpha,k+1}\left(-\lambda x^{\alpha}\right)$$

$$\therefore y(x) = \int_0^x (x-t)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda(x-t)^{\alpha}\right)h(t)dt + \sum_{k=0}^{n-1} E_{\alpha,k+1}\left(-\lambda x^{\alpha}\right)$$

2.3. The Composite Fractional Relaxation Equation

The computation of the solution of the following linear FDEs examined in this paper:

$$\frac{d^m u}{dt^m} - a\frac{d^{\alpha}u}{dt^{\alpha}} - bu = f(t), \quad t > 0, m-1 < \alpha \le m \quad ....(2.1)$$

subject to the initial conditions

$u^j(0)=c_j$, j=0,1,....., m-1

Where, C_j, j=0,1,....., m-1are undefined parameters and u(t) is assumed to be a causal connection of time, i.e., vanishing for t < 0. In the Caputo sense, fractional derivatives are taken into consideration. A parameter that describes the order of the fractional derivative is present in the actual response expression, and it can be changed to produce different answers. In the cases where we refer to the composite fractional oscillation equation, we use Eq. (2.1) for the composite fractional relaxation. {$0<\alpha\le1$, m=1} and {$0<\alpha\le2$, m=2}, respectively.

The fractional derivative of f(x) in the Caputo sense is defined as

$$D_*^{\alpha} f(x) = J^{m-\alpha}D^m f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}f^{(m)}(t)dt, \dots \dots \dots (2.2)$$

For $m-1<\alpha\le m$, $m\in N$, $x>0$, $f\in C^m_{-1}$

2.4. The Variational Iteration Method

The basic concepts of the VIM are explained in [He, 1999a], along with information on how it can be applied to numerous types of equations with differential coefficients. We examine the FDE that follows

$$D_t^m u - aD_{*t}^{\alpha} - bu = f(t) \dots \dots \dots\dots\dots\dots (2.3)$$

where $D^m_t=d^m/(dt^m)$ and the fractional differential operator D_(*t)^m is defined as in Eq. (2.2), subject to the initial conditions (2.1). The practical modification for Eq. (2.3) can be taken as

$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(D_t^m u_n(\upsilon) - aD^{\alpha}_{*t}\tilde{u}_n(\upsilon) - b\tilde{u}_n(\upsilon) - f(\upsilon))\, d\upsilon$ … (2.4)

where k is a Lagrange multiplier, which can be found in the best possible way using variational theory. here $\tilde{u}_n$ and $D^{\alpha}_{*t}\tilde{u}_n$ are regarded as limited differences. We start with the first rough estimate.

$u_0 = c_0 + c_1 t + c_2 t^2 + ... + c_{m-1}t^{m-1}$ …........................… (2.5)

that the functional stationary ahead of, Observing $\delta\tilde{u}_n = 0$,

$\delta u_{n+1}(t) = \delta u_n(t) + \delta\int_0^t \delta(D_t^m u_n(\upsilon)-f(\upsilon))d\upsilon$……………(2.6)

Results in the following Lagrange multipliers are as follows:

λ=-1 for m=1

λ=υ-t for m=2

Then, for m = 1, we have the following expand formula:

$u_{n+1}(t)= u_n(t)- \int_0^t(D_t^1 u_n(\upsilon) - aD_{*t}^{\alpha}u_n(\upsilon) - bu_n(\upsilon)-f(\upsilon))d$ ...(2.7)

For m = 2, we obtain the following iteration formula:

$u_{n+1}(t)=u_n(t)+\int_0^t(\upsilon\text{-}T)(D_t^2 u_n(\upsilon)-aD_{*t}^{\alpha}u_n(\upsilon)-bu_n(\upsilon)-f(\upsilon))d\upsilon$ ....(2.8)

2.5. Fractional differential transform method

The idea of variations can be broadened to non-integer orders in several ways. In the Riemann–Liouville logic, the fractional differentiation is described as

$$D_{x_0}^p f(x) = \frac{1}{\Gamma(m-p)}\frac{d^m}{dx^m}\left[\int_{x_0}^x \frac{f(t)}{(x-t)^{1+p-m}}dt\right] \dots \dots \dots\dots (2.9)$$

for $-1\le q<m$, $m\in Z^+$, $x>x\_0$. The logical and ongoing function f(x) can be widened as follows through a fractal power series:

$f(x) = \sum_{l=0}^{\infty} F(l)(x-x_0)^{l/\alpha}$……………………………..… (2.10)

where α is the non-integer order and F(l) is the fractal difference transform of (x). Having the real-world applications that took place in numerous subsidiaries of science, the fractional starting conditions are not avails all the time, and it may not be known what their actual meaning is. Also, the solution in Eq. (2.9) should be changed to taking with non-fractional ordered starting conditions in Caputo sense (He, 1998) as follows:

$$D_{x_0}^p\left[f(x) - \sum_{l=0}^{m-1}\frac{1}{l!}(x-x_0)^l f^{(l)}(0)\right] = \frac{1}{\Gamma(m-p)}\frac{d^m}{dx^m}\left\{\int_0^x\left[\frac{d(t)-\sum_{l=0}^{m-1}\frac{1}{l!}(t-x_0)^l f^{(l)}(0)}{(x-t)^{1+p-m}}\right]dt\right\}$$

…………….(2.11)

The modification of the starting points is defined below since they are applied to the variations of numerical order:

$$\begin{cases} \text{IF } l/_{\alpha}\in Z^+, \frac{1}{\left(\frac{l}{\alpha}\right)!}\left[\frac{d^{\frac{l}{\alpha}}f(x)}{dx^{\frac{l}{\alpha}}}\right]_{x=x_0} \\ \text{IF } l/_{\alpha}\notin Z, 0 \end{cases} \quad \dots\dots\dots\dots\dots\dots\dots\dots (2.12)$$

When l=0,1,2,....,(nα-1)

where, n is the number of FDE taken. Using Eqs. (2.1) and (2.2), the theorems of FDTM, are presented next.

Theorem 1:

If $f(x)=(x-x_0)^q$, then F(l)=δ(l-αq) where, $\delta(l)= \begin{cases}1\ if\ l=0\\0\ if\ l\neq 0\end{cases}$

Derivation:

For f(x), the statement that follows can be expressed with regard to of the dirac-delta operates as

$$f(x) = \sum_{l=0}^{\infty} \delta(l - \alpha q)(x - x_0)^{l/\alpha}$$

From the definition of transform, following expression can be obtained:

F(l) = δ(l - αq)

Theorem 2:

If $f(x) = D_{x_0}^{p}[g(x)]$, then $F(k) = \frac{\Gamma(p+1+l/\alpha)}{\Gamma(1+l/\alpha)} G(l+\alpha p)$

Proof:

Through the use of Eq. (2.11)The fractal decomposition of g(x) in the Caputo sense can be expressed this way:

$$D_{x_0}^{p}[g(x)] = \frac{1}{\Gamma(m-p)} \frac{d^m}{dx^m}\left\{\int_0^x \left[\frac{g(t) - \sum_{l=0}^{m-1} \frac{1}{l!}(t-x_0)^l g^{(l)}(0)}{(x-t)^{1+p-m}}\right] dt\right\}, m-1 \le p < m$$

Making use of Eqs. (2.10) aa well as (2.12), we get

$$D_{x_0}^{p}[g(x)] = \frac{1}{\Gamma(m-p)} \frac{d^m}{dx^m}\left[\int_{x_0}^{x} \frac{\sum_{l=0}^{\infty} G(k)(t-x_0)^{l/\alpha} - \sum_{k=0}^{q\alpha-1} G(k)(t-x_0)^{\frac{k}{\alpha}}}{(x-t)^{1+p-m}} dt\right]$$

$$= \frac{1}{\Gamma(m-p)} \sum_{l=p\alpha}^{\infty} G(k) \frac{d^m}{dx^m}\left[\int_{x_0}^{x} \frac{(t-x_0)^{l/\alpha}}{(x-t)^{1+p-m}} dt\right]$$

$$= \sum_{l=p\alpha}^{\infty} \frac{\Gamma(1+l/\alpha)}{\Gamma(1-p+l/\alpha)} G(l)(x-x_0)^{\frac{l}{\alpha}-p}$$

beginning this series' index from k = 0, we have

$$F(x) = \sum_{k=0}^{\infty} \frac{\Gamma(p+1+l/\alpha)}{\Gamma(1+l/\alpha)} G(l+\alpha p)(x-x_0)^{\frac{k}{\alpha}}$$

According to the parameters of transformation in Eq. (2.10), The calculation that follows results in:

$$F(k) = \frac{\Gamma(p+1+l/\alpha)}{\Gamma(1+l/\alpha)} G(l+\alpha p)$$

3. Results and Discussions

Here we look at a single instance to provide a clear overview of the methodology as a numerical element. To make a computational instance, we utilize the VIM and FDTM on those instances.

Example:

Consider the composite fractional oscillation equation:

$d^2u/dt^2 - a(d^\alpha u/dt^\alpha) - bu = 8$, $t>0$, $1<\alpha\le 2$ …………..(3.1)

For the initial conditions:

$u(0) = 0, u'(0)=0$ ..……………………………… (3.2)

In view of (3.1), the calculation for repetitions for (2.8) is given by

$u_{n+1}(t)=u_n(t)+\int_0^t(\upsilon-t)(D^2u_n(\upsilon)-aD_{*t}^{\alpha}u_n(\upsilon)-bu_n(\upsilon)-8)d\upsilon$ ...(3.3)

Further, we look for the following assumptions:

$u_0(t)=0$, $u_1(t)=4t^2$

$$u_0(t) = 0, u_1(t) = 4t^2$$

$$u_2(t) = 4t^2 + \frac{1}{3}bt^4 + \frac{8a}{\Gamma(5-\alpha)} t^{4-\alpha}$$

$$u_3(t) = 4t^2 + \frac{1}{3}bt^4 + \frac{8a}{\Gamma(5-\alpha)} t^{4-\alpha} + \frac{b^2}{90} t^6 + \frac{16ab}{\Gamma(7-\alpha)} t^{6-\alpha} + \frac{8a^2}{\Gamma(7-2\alpha)} t^{6-2\alpha} \quad ...(3.4)$$

and so forth. The remaining elements of the reiteration calculation (3.4) can also be gathered in the same way.

Taking a=b=-1 and using theorems 1 and theorem 2, the formula ahead of can be transferred as below

$$U(l+2\beta) = -\frac{\Gamma(\alpha+1+l/\beta)U(l+\beta\alpha) + \Gamma(1+l/\beta)[U(l) - 8\delta(l)]}{\Gamma(3+l/\beta)} \quad ...(3.5)$$

where β is the undefined fractal value. The terms in Eq. (3.1) can be transformed by using Eq. (2.12) as follows:

U(l)=0 for l=0,1,.....,2β-1 ………(3.6)

When used Eqs. (3.5) and (3.5), U(k) for k = 2β, 2β + 1,...,n is computed and using the inverse transforming process in Eq. (3.2), u(t) is computed for different values of α. chart 1 provides quantitative outcomes for comparison where the term "accurate solution" refers to the closed form series solution given in (Hilfer, 2000). These findings show that FDTM can achieve six digits of accuracy by using N = 10 terms.

The accurate solution of equation (3.13) is given by

$$u(t) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} \frac{8a^j b^{k-j}}{\Gamma(2k-j\alpha+3)} t^{2k-j\alpha+2} \quad ...(3.7)$$

Taking 10 terms after expanding, we get

$$u(t) = \frac{8t^{2-j\alpha}}{(2-j\alpha)!} + \frac{8\left(\frac{a}{b}+1\right) b\, t^{4-j\alpha}}{(4-j\alpha)!} + \frac{8\left(\frac{a}{b}+1\right)^2 b^2 t^{6-j\alpha}}{(6-j\alpha)!} + \frac{8\left(\frac{a}{b}+1\right)^3 b^3 t^{8-j\alpha}}{(8-j\alpha)!} + \frac{8\left(\frac{a}{b}+1\right)^4 b^4 t^{10-j\alpha}}{(10-j\alpha)!} + \frac{8\left(\frac{a}{b}+1\right)^5 b^5 t^{12-j\alpha}}{(12-j\alpha)!} + \frac{8\left(\frac{a}{b}+1\right)^6 b^6 t^{14-j\alpha}}{(14-j\alpha)!} + \frac{8\left(\frac{a}{b}+1\right)^7 b^7 t^{16-j\alpha}}{(16-j\alpha)!} + \frac{8\left(\frac{a}{b}+1\right)^8 b^8 t^{18-j\alpha}}{(18-j\alpha)!} + \frac{8\left(\frac{a}{b}+1\right)^9 b^9 t^{20-j\alpha}}{(20-j\alpha)!}$$

Putting *a=-1, b=-1* we get

$$U(t) = -\frac{16t^{4-j\alpha}}{(4-j\alpha)!} + \frac{32t^{6-j\alpha}}{(6-j\alpha)!} - \frac{64t^{8-j\alpha}}{(8-j\alpha)!} + \frac{128t^{10-j\alpha}}{(10-j\alpha)!} - \frac{256t^{12-j\alpha}}{(12-j\alpha)!} + \frac{512t^{14-j\alpha}}{(14-j\alpha)!} - \frac{1024t^{16-j\alpha}}{(16-j\alpha)!} + \frac{2048t^{18-j\alpha}}{(18-j\alpha)!} - \frac{4096t^{20-j\alpha}}{(20-j\alpha)!} + \frac{8192t^{22-j\alpha}}{(22-j\alpha)!}$$

After putting value α=1.5, we get the exact value.

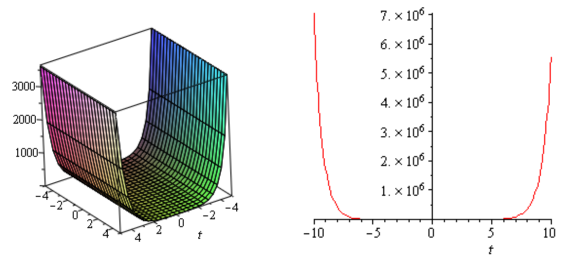

Figure 1. 3-D and 2-D plot of equation (3.7) when a= -1, b= -1 and α=1.5

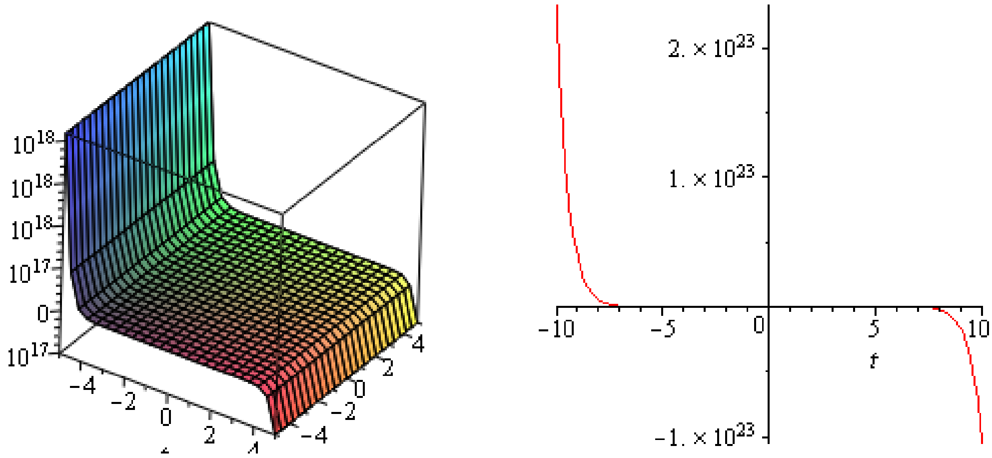

Figure 2. 3-D and 2-D plot of equation (3.7) when a= -5, b= -7 and α=1.5

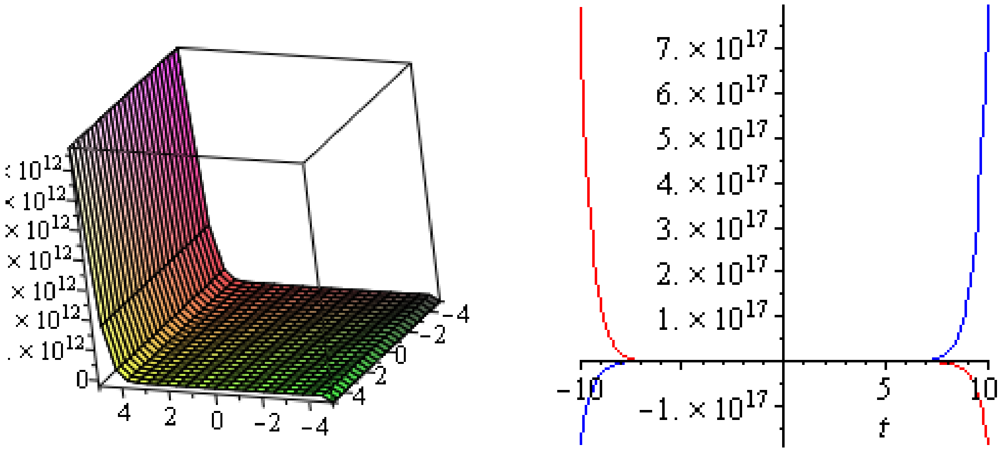

Figure 3. 3-D and 2-D plot of equation (3.7) when a=3, b=-11 and α=1.5

Table 1. Comparison of VIM and FDTM methods

α=1.5			
t	$\mathbf{u}_{FDTM}$	$\mathbf{u}_{VIM}$	$\mathbf{u}_{exact}$
0.0	0.00000000	0.00000000	0.00000000
0.1	0.03350759	0.03647845	0.03350723
0.2	0.12522237	0.14064078	0.12522196
0.3	0.26760933	0.30748531	0.26760904
0.4	0.45543589	0.53328417	0.45543567
0.5	0.68433621	0.81475709	0.68433583
0.6	0.95039364	1.14884054	0.95039339
0.7	1.24995953	1.53257119	1.24995924
0.8	1.57955733	1.96303357	1.57955709
0.9	1.93583279	2.43733178	1.93583254
1.0	2.31552656	2.95256745	2.31552608

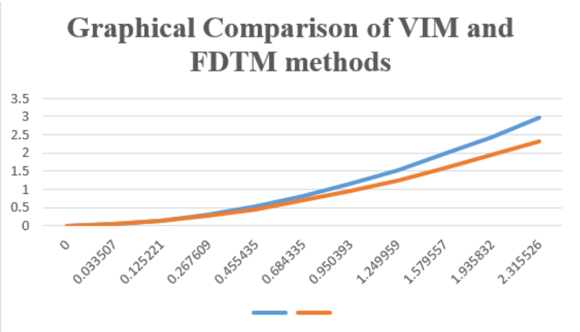

Figure 4. Graphical comparison shown of table 1

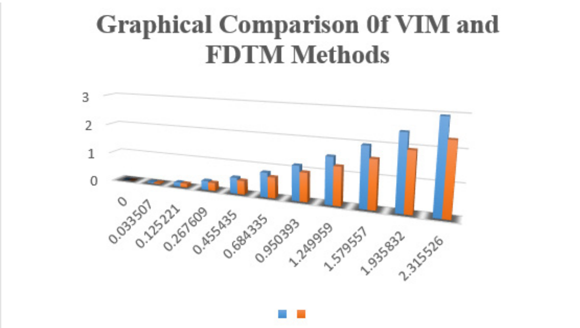

Figure 5. Graphical diagram comparison shown of table 1

Table 2. Error analysis of VIM and FDTM method

t	Error u_{VIM}	Error u_{FDTM}
0.0	0	0
0.1	-0.00297122	-3.6E-07
0.2	-0.01541882	-4.1E-07
0.3	-0.03987627	-2.9E-07
0.4	-0.0778485	-2.2E-07
0.5	-0.13042126	-3.8E-07
0.6	-0.19844715	-2.5E-07
0.7	-0.28261195	-2.9E-07
0.8	-0.38347648	-2.4E-07
0.9	-0.50149924	-2.5E-07
1.0	-0.63704137	-4.8E-07

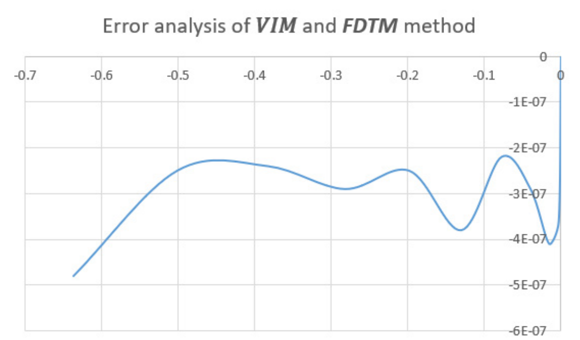

Figure 6. Graphical error comparison shown of table 2

from the number based results in chart 1 and 2 indicates the fact that the fractional differential transform method gives a highly accurate solution.

4. Conclusions

This paper presents the successful incorporation of the Fractional Differential Transform Method (FDTN) and The Variational Iteration Method (VIM) for solving nonlinear composite fractional oscillation equation. In summary, the FDTN and the VIM could be viewed as a nice improvement over current numerical techniques and could have a wide range of uses. The current analysis demonstrates how to solve fractional order linear difference equations using the VIM and the FDTM. When applied to actual physical problems, they offer faster-than-realistic series solutions. Notably, the two approaches have the advantage that, in the nonlinear stochastic case, those do not require closest approximation, general assumptions, linearization, discretization, perturbation, or practically implausible assumptions.

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